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International Journal of Solids and Structures 43 (2006) 1422–1436
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Nonlinear behaviour of piezoceramics under weak electric fields Part-I: 3-D finite element formulation

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Received 17 November 2004; received in revised form 13 June 2005
Available online 30 August 2005

Abstract

Piezoceramic materials exhibit different types of nonlinearities under different combinations of electric and mechanical fields. When excited near resonance in the presence of weak electric fields, they exhibit typical nonlinearities similar to a Duffing oscillator such as jump phenomena and presence of superharmonics in the response spectra. In order to model such nonlinearities, a nonlinear electric enthalpy density function (using quadratic and cubic terms) valid for a general 3-D piezoelectric continuum has been proposed in this work. Linear (i.e. proportional) and nonlinear damping models have also been proposed. The coupled nonlinear finite element equations have been derived using variational formulation. The classical linearization technique has been used to derive the linearized stiffness and damping matrices which helps in assembling the nonlinear matrices and solution of resulting nonlinear equation. The general 3-D finite element formulation is discussed in this paper. In a companion paper by Samal et al., numerical results on various typical examples are shown to match very well with the experimental observations.

Keywords: Finite element method; Piezoceramic; Nonlinear modeling; Jump phenomena; Weak electric fields; Structural mode

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.06.084

1. Introduction

Smart materials, especially piezoceramics find a wide range of applications viz., active vibration control, shape control, health monitoring of structures (Crawley, 1994; Rao and Sunar, 1994). Piezoceramics have been extensively used in recent years for actuators and sensors applications because of their high electromechanical coupling coefficients. A special class of these actuators, known as ultrasonic transducers, are driven at resonance frequency. It is well known that the mechanical and electrical responses of a piezoelectric material are coupled. When the applied electric field is low and the strains are also low, the behaviour of piezoceramics is almost linear. But, the piezoelectric materials exhibit a wide range of nonlinear effects under both high and low electric fields when operated in the presence of high and low stress fields. The nonlinear behaviours under high and low electric fields are different in some aspects. For example, the nonlinearity observed under high electric fields is typical hysteresis behaviour between electric field and polarization, electric field and strain such as dielectric and butterfly hysteresis etc. which occurs because of domain switching and the associated ferroelectric nonlinearities. The nonlinearities observed under weak electric fields are jump phenomena, dependence of resonance frequency on vibration amplitude, presence of superharmonics in the response spectra etc. The problems associated with piezoactuators showing the jump phenomena can be excessive heat generation, mechanical break down and instability when operating in the resonant mode etc. The different nonlinear behaviours of piezoactuators can also affect other applications such as the active shape and vibration control of structures (Sun et al., 2004; Zhou and Tzou, 2000).

There are many models for nonlinearities under strong electric fields (i.e., hysteresis), which are mainly based on classical Preisach model and its modified versions (Zhou and Chattopadhyay, 2001; Yu et al., 2002). Hwang and McMeeking (1998) have used a domain wall switching theory to explain the hysteresis in ferroelectric polycrystals and developed a finite element model based on the above theory. However, there are very few research works existing in the literature which deal with the nonlinear effects under weak electric fields. Beige and Schmidt (1982) first observed these nonlinear effects while conducting experiments on longitudinal vibrations of piezorods. They developed analytical models for these nonlinearities using higher order quadratic and cubic terms in the energy expression contributed by elastic, piezoelectric and dielectric continuum. von Wagner and Hagedorn (2002) and von Wagner (2003) modeled nonlinear behaviour of the typical PZT based piezoceramics excited using d_{31} and d_{33} effects. In these papers, the softening behaviour of the material is attributed to the nonlinear dependence of the Young's modulus and the piezoelectric coefficient on the mechanical and electrical field variables. They derived the electric enthalpy density functions for the 1-D continuum. The governing 1-D nonlinear equation was derived using variational formulation (Hamilton's principle) and solved using Rayleigh–Ritz method.

Benjeddou (2000) has given a detailed review of different piezoelectric finite element models that have been developed to model adaptive structural elements. Several authors have used FEM to model various linear piezoelectric material systems (Allik and Hughes, 1970; Allik et al., 1974; Tzou and Tseng, 1990; Tzou and Tseng, 1991; Tsung and Charles, 1993; Simkovics et al., 1999a,b). Saravanos et al. (1997) developed a layer-wise FE model for the dynamic analysis of piezoelectric composite plates. Recently, Wang (2004) developed a FE model for static and dynamic analysis of piezoelectric bimorphs. Wang et al. (2004) studied the dynamic stability behaviour of FE models for piezocomposite plates.

It is observed that a generalized FE model taking into consideration the nonlinearity of piezocontinuum under weak electric fields similar to a Duffing oscillator is lacking. A generalized 3-D finite element model for the above nonlinearity is proposed in this work. The salient features of this work are given as below.

(a) Development of a generalized 3-D nonlinear electric enthalpy density function taking into consideration of quadratic and cubic terms in the mechanical, dielectric and piezoelectric domain.
(b) Development of generalized 3-D virtual work to model nonlinear damping in the material considering dissipative energy terms in the mechanical, dielectric and piezoelectric domain.

(c) Linearization of the resulting nonlinear finite element equations (after application of variational principle) to obtain the coupled equations suitable for assembling and solution using Newmark-β method.

The experimentally observed behaviours such as jump phenomena, nonlinear softening, dependence of resonance frequency on excitation amplitude and presence of superharmonics in the response spectra etc. need to be captured in the model and these are the motivations of this work. The goal of capturing the nonlinearities in a model has been achieved here by inclusion of quadratic and cubic order terms in the generalized electric enthalpy density as well as generalized virtual work (by nonlinear damping forces). The proposed generalized 3-D nonlinear finite element model is extremely useful for analyzing the complex piezoceramic structures where closed form solution cannot be readily derived. Besides, results of 3-D FE model can also serve as a benchmark to check the accuracy of various 1-D or 2-D analytical models.

2. Constitutive equations of the piezoelectric continuum

2.1. Linear electric enthalpy density function

The electric enthalpy density function H is generally used to derive the governing equations of the coupled piezoelectric continuum and for the linear piezoelectric behaviour, it is given as (IEEE standard, 1988)

$$H_{\text{lin}} = \frac{1}{2} C^{\text{E}}_{ijkl} S_{ij} S_{kl} - e_{kij} E_k S_{ij} - \frac{1}{2} \varepsilon^{\text{S}}_{ij} E_i E_j \tag{1}$$

where C^{E}_{ijkl} is the fourth order elastic tensor under constant electric field, e_{kij} is the 3rd order piezoelectric tensor, $\varepsilon^{\text{S}}_{ij}$ is the second order dielectric tensor, S_{ij} is the strain tensor and E_i is the electric field vector. The second Piola–Kirchoff stress tensor T_{ij} and the electric displacement vector D_i can be derived from H using the expressions $T_{ij} = \partial H/\partial S_{ij}$ and $D_i = -\partial H/\partial E_i$ respectively. It can be easily seen that the expressions for T_{ij} and D_i are coupled with the secondary field variables S_{ij} (strain tensor) and E_i (electric field vector) respectively. S_{ij} is expressed in terms of mechanical displacement vector u_i as $S_{ij} = (u_{i,j} + u_{j,i})/2$ and E_i is expressed in terms of ϕ as $E_i = -\phi_{,i}$. Keeping in mind the fact that stress and strain tensors are symmetric, H_{lin} can be simplified and written in terms of second order tensors as (Maugin, 1985)

$$H_{\text{lin}} = \frac{1}{2} C^{\text{E}}_{ij} S_i S_j - e_{ij} E_i S_j - \frac{1}{2} \varepsilon^{\text{S}}_{ij} E_i E_j \tag{2}$$

2.2. Proposed nonlinear electric enthalpy density function

In order to model the nonlinearities in a coupled piezoelectric medium, the linear electric enthalpy density function is modified in the following way. It was experimentally observed (von Wagner and Hagedorn, 2002; Dave, 2002; von Wagner, 2003) that second and third order harmonics were present in the response spectra of PZT piezoceramic plates and rod geometries. They also observed jump phenomena in both displacement and current response for materials with low damping. This jump phenomenon is a characteristic of a system with cubic stiffness in all the domains i.e., mechanical, dielectric and piezoelectric. Hence, cubic terms were added in the generalized electric enthalpy density function for all of the above domains. As second and third order harmonics were observed in the response spectra of PZT piezoceramic plates and rod geometries, it was envisaged to develop a generalized nonlinear electric enthalpy density function including both quadratic and cubic terms. Hence a generalized expression for nonlinear electric enthalpy density function valid for a 3-D piezoelectric continuum is proposed as follows:

$$H_{\mathrm{nonl}} = \frac{1}{2}C^{\mathrm{E}}_{ij}S_iS_j - e_{mi}E_mE_i - \frac{1}{2}\varepsilon^{\mathrm{S}}_{mn}E_mE_n + \frac{1}{3}C^{\mathrm{E}}_{ijk}S_iS_jS_k + \frac{1}{4}C^{\mathrm{E}}_{ijkl}S_iS_jS_kS_l - \frac{1}{3}\varepsilon^{\mathrm{S}}_{mno}E_mE_nE_o - \frac{1}{4}\varepsilon^{\mathrm{S}}_{mnop}E_mE_nE_oE_p - \frac{1}{2}e_{mij}E_mS_iS_j - \frac{1}{2}e^{*}_{mni}E_mE_nS_i - \frac{1}{3}e^{**}_{mijk}E_mS_iS_jS_k - \frac{1}{4}e^{***}_{mnij}E_mE_nS_iS_j - \frac{1}{3}e^{****}_{mnoi}E_mE_nE_oS_i \quad (3)$$

where the coefficients with 3 and 4 number of subscript indices are higher order quadratic and cubic coefficients respectively. The superscripts (*, ** etc.) on coefficient e_{mnij} are used to differentiate between different nonlinear piezoelectric coefficients. In order to understand the logical extension of the nonlinear electric enthalpy function for the 3-D case from the linear electric enthalpy function, the reader is referred to Appendix A.1 where the derivation of H_{nonl} has been explained corresponding to a 1-D beam example (von Wagner et al., 2001).

A matrix version of the above expression for H_{nonl} suitable for finite element formulation is given as follows:

$$H_{\mathrm{nonl}} = \frac{1}{2}\{S\}^{\mathrm{T}}[C]\{S\} - \{E\}^{\mathrm{T}}[d][C]\{S\} - \frac{1}{2}\{E\}^{\mathrm{T}}[[\varepsilon^{\mathrm{T}}] - [d][C][d]^{\mathrm{T}}]\{E\} + \frac{1}{3}\{S\}^{\mathrm{T}}[C_1]\{S^2\} + \frac{1}{4}\{S^2\}^{\mathrm{T}}[C_{21}]\{S^2\} + \frac{1}{4}\{S\}^{\mathrm{T}}[C_{22}]\{S^3\} - \frac{1}{2}\{E\}^{\mathrm{T}}[\gamma_{11}]\{S^2\} - \frac{1}{2}\{E^2\}^{\mathrm{T}}[\gamma_{12}]\{S\} - \frac{1}{3}\{E\}^{\mathrm{T}}[\upsilon_1]\{E^2\} - \frac{1}{3}\{E\}^{\mathrm{T}}[\gamma_{21}]\{S^3\} - \frac{1}{2}\{E^2\}^{\mathrm{T}}[\gamma_{22}]\{S^2\} - \frac{1}{3}\{E^3\}^{\mathrm{T}}[\gamma_{23}]\{S\} - \frac{1}{4}\{E\}^{\mathrm{T}}[\upsilon_{21}]\{E^3\} - \frac{1}{4}\{E^2\}^{\mathrm{T}}[\upsilon_{22}]\{E^2\} \quad (4)$$

where $\{S\}$ is the strain vector, $[C]$ is linear elasticity matrix, $[d]$ is the linear piezoelectric coefficient matrix, $\{E\}$ is the electric field vector, $[\varepsilon^{\mathrm{T}}]$ is the dielectric coefficient matrix, $[C_1]$ is quadratic elasticity matrix, $[C_{21}]$ and $[C_{22}]$ are cubic elasticity matrices at constant electric field respectively. The superscript E and S (for the matrices at constant electric field and constant strain) are omitted for clarity. $[\gamma_{11}]$ and $[\gamma_{12}]$ are quadratic piezoelectric matrices, $[\gamma_{21}]$, $[\gamma_{22}]$ and $[\gamma_{23}]$ are cubic piezoelectric matrices, $[\upsilon_0] = [[\varepsilon^{\mathrm{T}}] - [d][C][d]^{\mathrm{T}}]$, $[\upsilon_1]$, $[\upsilon_{21}]$ and $[\upsilon_{22}]$ are linear, quadratic and cubic dielectric matrices at constant strain field S respectively. The vectors with superscripts 2 and 3 are defined as the vectors with square and cube of individual terms of the vectors. For example, the vector $\{S^2\}$ is written as $[S_1^2, S_2^2, S_3^2, S_4^2, S_5^2, S_6^2]^{\mathrm{T}}$. The first three terms of H_{nonl} correspond to the linear electric enthalpy density function. Using this expression of H_{nonl}, the FE matrices have been derived in the next section using variational formulation.

3. Variational formulation

The equations of motion for a piezoelectric continuum can be derived from Hamilton's principle, in which the Lagrangian and the virtual work are properly adapted to include the electrical, mechanical as well as the coupled electro-mechanical terms. The potential energy density of a piezoelectric material includes contributions from the strain energy and from the electrostatic energy. Hamilton's principle can be written as

$$\delta\int_{t_0}^{t_1}\left(\int_V L\,\mathrm{d}V\right)\mathrm{d}t + \int_{t_0}^{t_1}\delta W\,\mathrm{d}t = 0 \quad (5)$$

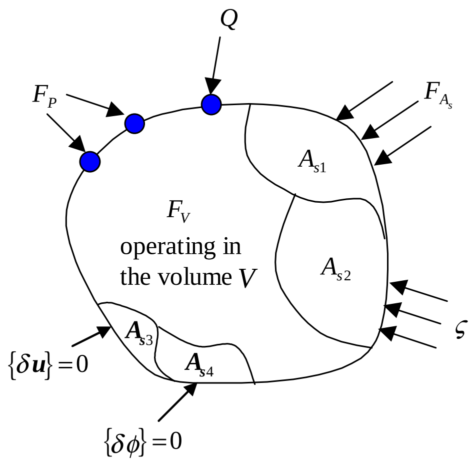

Fig. 1. Generalized mechanical and electrical forces of various types acting in the domain V.

where t_0 and t_1 define the time interval, L is the Lagrangian and defined in terms of the kinetic energy density T_{ke} and electrical enthalpy density H as $L = (T_{\mathrm{ke}} - H)$, δw is the virtual work done by external mechanical and electrical forces. The kinetic energy is given as

$$T_{\mathrm{ke}} = \frac{1}{2}\rho\{\dot{u}\}^{\mathrm{T}}\{\dot{u}\}. \tag{6}$$

where ρ is the mass density, $\{u\}$ is the generalized displacement field and $\{\dot{u}\}$ is the generalized velocity field. The extended electric enthalpy density H_{nonl} is given in Eq. (4). The virtual work done by the external mechanical forces F and the applied electric charges Q for an arbitrary variation of the displacement field $\{\delta u\}$ and the electrical potential $\{\delta\phi\}$ both satisfying the essential boundary conditions (i.e., $\{\delta u\} = 0$ on surface A_{s3} and $\{\delta\phi\} = 0$ on surface A_{s4} [as shown in Fig. 1]) can be written as

$$\delta W = \int_V \{\delta u\}^{\mathrm{T}}\{F_V\}\,\mathrm{d}V + \int_{A_{s1}} \{\delta u\}^{\mathrm{T}}\{F_{A_s}\}\,\mathrm{d}A_s + \{\delta u\}^{\mathrm{T}}\{F_P\} - \int_{A_{s2}} \delta\phi\zeta\,\mathrm{d}A_s - \delta\phi Q + \delta W_{\mathrm{D}} \tag{7}$$

where $\{F_V\}$ is the applied body force vector, $\{F_{A_s}\}$ is the applied surface force vector (defined on the surface A_{s1}), $\{F_P\}$ is the applied point load vector, ϕ is the electric potential, ζ is the surface charge brought on surface A_{s2}, Q is the applied concentrated electric charge and δW_{D} is the virtual work done by damping forces.

Now, substituting the nonlinear electric enthalpy density function H_{nonl} and kinetic energy density T_{ke} from Eqs. (4) and (6) in Eq. (5), we get the following expression:

$$- \int_{t_0}^{t_1} \int_V [\rho\{\delta u\}^{\mathrm{T}}\{\ddot{u}\}]\,\mathrm{d}V\,\mathrm{d}t$$

$$- \int_{t_0}^{t_1} \int_V \{\delta S\}^{\mathrm{T}} \begin{bmatrix} [C]\{S\} - [C]^{\mathrm{T}}[d]^{\mathrm{T}}\{E\} - [\mathrm{diag}(\{S\})][\gamma_{11}]^{\mathrm{T}}\{E\} - \frac{1}{2}[\gamma_{12}]^{\mathrm{T}}\{E^2\} \\ +[\mathrm{diag}(\{S\})][C_{21}][\mathrm{diag}(\{S\})]\{S\} + \frac{1}{4}[C_{22}]\{S^3\} + \frac{3}{4}[\mathrm{diag}(\{S^2\})][C_{22}]^{\mathrm{T}}\{S\} \\ +\frac{1}{3}[C_1]\{S^2\} + \frac{2}{3}[\mathrm{diag}(\{S\})][C_1]^{\mathrm{T}}\{S\} - [\mathrm{diag}(\{S^2\})][\gamma_{21}]^{\mathrm{T}}\{E\} \\ -[\mathrm{diag}(\{S\})][\gamma_2 2]^{\mathrm{T}}\{E^2\} - \frac{1}{3}[\gamma_{23}]^{\mathrm{T}}\{E^3\} \end{bmatrix} \mathrm{d}V\,\mathrm{d}t$$

$$-\int_{t_0}^{t_1}\int_V \{\delta E\}^{\mathrm{T}}\begin{bmatrix}[d][C]\{S\}+[\upsilon_0]\{E\}+\frac{1}{2}[\gamma_{11}]\{S^2\}\\ +[\mathrm{diag}(\{E\})][\gamma_{12}]\{S\}+\frac{1}{3}[\gamma_{13}]\{S^2\}+\frac{2}{3}[\mathrm{diag}(\{E\})][\gamma_{13}]^{\mathrm{T}}\{E\}\\ +\frac{1}{3}[\gamma_{21}]\{S^3\}+[\mathrm{diag}(\{E\})][\gamma_{22}]\{S^2\}+[\mathrm{diag}(\{E^2\})][\gamma_{23}]\{S\}\\ +\frac{1}{4}[\upsilon_1]\{E^3\}+\frac{3}{4}[\mathrm{diag}(\{E^2\})][\upsilon_1]^{\mathrm{T}}\{E\}+[\mathrm{diag}(\{E\})][\upsilon_2][\mathrm{diag}(\{E\})]\{E\}\end{bmatrix}\mathrm{d}V\,\mathrm{d}t$$
$$+\int_{t_0}^{t_1}\int_V\{\delta u\}^{\mathrm{T}}\{F_V\}\,\mathrm{d}V\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{A_{S1}}[\{\delta u\}^{\mathrm{T}}\{F_{A_s}\}]dA_s\,\mathrm{d}t+\int_{t_0}^{t_1}\{\delta u\}^{\mathrm{T}}\{F_{\mathrm{P}}\}\,\mathrm{d}t-\int_{t_0}^{t_1}\int_{A_{s2}}\delta\phi\zeta\,\mathrm{d}A_s\,\mathrm{d}t$$
$$-\int_{t_0}^{t_1}\delta\phi Q\,\mathrm{d}t-\int_{t_0}^{t_1}[\delta W_{\mathrm{D}}]\,\mathrm{d}t=0 \tag{8}$$

where the matrix with 'diag' refers to a diagonal matrix with the main diagonal terms being the terms of the vector it contains. For deriving the expression for the virtual work done by damping forces (i.e., δW_{D}), two types of models are considered and are given below.

3.1. Proportional damping formulation

The virtual work done by viscous damping forces is given as

$$\delta W_{\mathrm{D}}=\int_V\{\delta u\}^{\mathrm{T}}[C_{\mathrm{damp}}]\{\dot u\}\,\mathrm{d}V, \tag{9}$$

where $[C_{\mathrm{damp}}]$ is the proportional damping coefficient matrix and is expressed in terms of mass $[M]$ and stiffness $[K]$ matrices with the help of constants α and β as

$$[C_{\mathrm{damp}}]=\alpha[M]+\beta[K]. \tag{10}$$

3.2. Nonlinear damping formulation

3.2.1. Virtual work δW_D—linear case

The virtual work expression for linear damping (i.e., dissipative energy due to damping) has been formulated considering work done by the linear viscous damping forces and is represented as (Ikeda, 1990; von Wagner, 2004)

$$\delta W_{\mathrm{D}}=\delta\int_V\{S\}^{\mathrm{T}}[C_d^{(0)}]\{\dot S\}\,\mathrm{d}V \tag{11}$$

The above expression for δW_{D} can be extended by incorporating other linear sources of damping, i.e. linear piezoelectric and linear dielectric damping matrices respectively (Ikeda, 1990; von Wagner, 2004). So, δW_{D} can be written as

$$\delta W_{\mathrm{D}}=\delta\int_V[\{S\}^{\mathrm{T}}[C_d^{(0)}]\{\dot S\}-\{S\}^{\mathrm{T}}[\gamma_{0d}]\{\dot E\}-\{\dot S\}^{\mathrm{T}}[\gamma_{0d}]\{E\}-\{E\}^{\mathrm{T}}[\upsilon_{0d}]\{\dot E\}]\,\mathrm{d}V \tag{12}$$

where $[\gamma_{0d}]$ and $[\upsilon_{0d}]$ are the linear piezoelectric and dielectric damping matrices, respectively.

3.2.2. Virtual work δW_D—nonlinear case

The basic linear expression for virtual work [Eq. (12)] can now be extended to combine the linear and nonlinear form by including quadratic and cubic damping terms in the same manner as done for the nonlinear electric enthalpy density function (von Wagner, 2004). This is done in order to obtain a generalized formulation for damping where the damping constant would include the effect of damping in mechanical, piezoelectric and dielectric domains. Also, the damping constants (experimentally determined from half power method) were observed to depend nonlinearly on the electric field (Samal, 2003). The experimental displacement-frequency responses of many piezoelectric structures were found to be non-symmetric (Samal, 2003). The nonsymmetric nature of the frequency response curve arises mainly because of nonlinear softening of the piezoelectric continuum which causes it to behave as a softening spring-damper system. Hence, the backbone curve of the response bends towards lower frequency. Also, the presence of cubic terms in the stiffness and damping expressions causes the jump phenomenon. The above nonlinear phenomenon can be captured in the model by considering higher order terms in the work done by damping forces from all sources.

Considering the effect of quadratic and cubic terms in all the fields (i.e., mechanical, dielectric and piezoelectric), the modified expression for nonlinear $\delta W_{\mathrm{D_{nonl}}}$ can be written as below:

$$\delta W_{\mathrm{D_{nonl}}} = \delta \int_V \left[\begin{array}{l} \{S\}^{\mathrm{T}}[C_d^{(0)}]\{\dot{S}\} - \{S\}^{\mathrm{T}}[\gamma_{0d}]\{\dot{E}\} - \{\dot{S}\}^{\mathrm{T}}[\gamma_{0d}]\{E\} - \{E\}^{\mathrm{T}}[\nu_{0d}]\{\dot{E}\} + \frac{1}{3}\{S\}^{\mathrm{T}}[C_d^{(1)}]\{(\dot{S}^2)\} \\ +\frac{1}{3}\{\dot{S}\}^{\mathrm{T}}[C_d^{(1)}]\{\dot{S}^2\} - \frac{1}{2}\{S^2\}^{\mathrm{T}}[\gamma_{1d}^{(1)}]\{\dot{E}\} - \frac{1}{2}\{(\dot{S}^2)\}^{\mathrm{T}}[\gamma_{1d}^{(1)}]\{E\} - \frac{1}{2}\{S\}^{\mathrm{T}}[\gamma_{1d}^{(2)}]\{(\dot{E}^2)\} \\ -\frac{1}{2}\{\dot{S}\}^{\mathrm{T}}[\gamma_{1d}^{(2)}]\{E^2\} + \frac{1}{4}\{S\}^{\mathrm{T}}[C_{d1}^{(2)}]\{(\dot{S}^3)\} + \frac{1}{4}\{\dot{S}\}^{\mathrm{T}}[C_{d1}^{(2)}]\{S^3\} + \frac{1}{2}\{S^2\}^{\mathrm{T}}[C_{d2}^{(2)}]\{(\dot{S}^2)\} \\ -\frac{1}{3}\{S^3\}^{\mathrm{T}}[\gamma_{2d}^{(1)}]\{\dot{E}\} - \frac{1}{3}\{(\dot{S}^3)\}^{\mathrm{T}}[\gamma_{2d}^{(1)}]\{E\} - \frac{1}{2}\{S^2\}^{\mathrm{T}}[\gamma_{2d}^{(2)}]\{(\dot{E}^2)\} - \frac{1}{2}\{(\dot{S}^2)\}^{\mathrm{T}}[\gamma_{2d}^{(2)}]\{E^2\} \\ -\frac{1}{3}\{S\}^{\mathrm{T}}[\gamma_{2d}^{(3)}]\{(\dot{E}^3)\} - \frac{1}{3}\{\dot{S}\}^{\mathrm{T}}[\gamma_{2d}^{(3)}]\{E^3\} - \frac{1}{3}\{E^2\}^{\mathrm{T}}[\nu_{1d}]\{\dot{E}\} - \frac{1}{3}\{(\dot{E}^2)\}^{\mathrm{T}}[\nu_{1d}]\{E\} \\ -\frac{1}{4}\{E^3\}^{\mathrm{T}}[\nu_{2d1}]\{\dot{E}\} - \frac{1}{4}\{(\dot{E}^3)\}^{\mathrm{T}}[\nu_{2d1}]\{E\} - \frac{1}{4}\{(\dot{E}^2)\}^{\mathrm{T}}[\nu_{2d2}]\{E^2\} - \frac{1}{4}\{E^2\}^{\mathrm{T}}[\nu_{2d2}]\{(\dot{E}^2)\} \end{array} \right] \mathrm{d}V \tag{13}$$

where $[C_d^{(1)}]$ is the quadratic elastic damping matrix, $[\gamma_{1d}^{(1)}]$, $[\gamma_{1d}^{(2)}]$, are quadratic piezoelectric damping matrices, $[\nu_{1d}]$ is the quadratic dielectric damping matrix, $[C_{d1}^{(2)}]$, $[C_{d2}^{(2)}]$ are cubic elastic damping matrices respectively. $[\gamma_{2d}^{(1)}]$, $[\gamma_{2d}^{(2)}]$ and $[\gamma_{2d}^{(3)}]$ are cubic piezoelectric damping matrices and ,are cubic dielectric damping matrices respectively. These matrices are also called *dissipative constant* matrices in the matrix equation of motion. It may also be noted here that the product of vectors inside a vector notation } implies that it is a resulting vector whose terms are products of the corresponding terms of two or more vectors. For example, $\{S\dot{S}\} = \{S_1\dot{S}_1, S_2\dot{S}_2, S_3\dot{S}_3, S_4\dot{S}_4, S_5\dot{S}_5, S_6\dot{S}_6\}$ where $\{S\}$ is a strain tensor.

The variation of W_{D} (i.e., δW_{D}) is given in Appendix A.2 which is used for deriving the FE equations of motion.

4. Derivation of nonlinear FE equations

The displacement field $\{u\}$ and the electric potential ϕ at any point in a finite element are related to the corresponding nodal values $\{u_i\}$ and $\{\phi_i\}$ by means of the element shape function matrices $[N_u]$ and $[N_\phi]$ as

$$\{u\} = [N_u]\{u_i\} \quad \text{and} \quad \phi = [N_\phi]\{\phi_i\} \tag{14}$$

The strain field vector and the electric field vector $\{E\}$ can be expressed in terms of $\{u_i\}$ and $\{\phi\}$ as

$$\{S\} = [B_u]\{u_i\} \quad \text{and} \quad \{E\} = [B_\phi]\{\phi_i\} \tag{15}$$

where $[B_U]$ and $[B_\phi]$ are derivatives of shape function matrices. Using these Eqs. (14) and (15) for the primary and secondary field variables, the FE equations are derived below here for different damping formulations (i.e., viscous proportional damping and nonlinear damping).

4.1. Derivation of finite element equations for the proportional damping formulation

Putting the expressions of $\{S\}$ and $\{E\}$ in the variational Eq. (8) with the expression of variation of virtual work for the proportional damping formulation δW_D from Eq. (9), collecting the terms with $\{\delta u_i\}^T$ and $\{\delta\phi_i\}^T$ and equating them to zero separately, we get the nonlinear finite element equations (for the current time step $(t+1)$) as

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}^{(t+1)} \begin{Bmatrix} \ddot{u}_i \\ \ddot{\phi}_i \end{Bmatrix}^{(t+1)} + \begin{bmatrix} K_{uu} & K_{u\phi} \\ K_{\phi u} & K_{\phi\phi} \end{bmatrix}^{(t+1)}_{\text{linear}} \begin{Bmatrix} u_i \\ \phi_i \end{Bmatrix}^{(t+1)} + \begin{Bmatrix} f_{m\_i} \\ f_{e\_i} \end{Bmatrix}^{(t+1)}_{\text{nonl}} = \begin{Bmatrix} f_i \\ g_i \end{Bmatrix}^{(t+1)} \tag{16}$$

This equation cannot be solved as it contains nonlinear mechanical and electrical force vectors (i.e., $\{f_{m\_i}\}_{\text{nonl}}$ *and* $\{f_{e\_i}\}_{\text{nonl}}$) which depend upon the solution vectors $\{u_i\}^{(t+1)}$ and $\{\phi_i\}^{(t+1)}$. This equation has been linearized following a classical linearization technique (Salinas et al., 1993). The coupled linearized FE equations are obtained as

$$[M]^{(t+1)}\{\ddot{u}_i\}^{(t+1)} + [C_{\text{damp}}]^{(t)}\{\dot{u}_i\}^{(t+1)} + [K_{uu}]^{(t)}\{u_i\}^{(t+1)} + [K_{u\phi}]^{(t)}\{\phi_i\}^{(t+1)} = \{f_i\}^{(t+1)} \tag{17}$$

$$[K_{\phi u}]^{(t)}\{u_i\}^{(t+1)} + [K_{\phi\phi}]^{(t)}\{\phi_i\}^{(t+1)} = \{g_i\}^{(t+1)} \tag{18}$$

where $[M]$ is the mass matrix, $[K_{uv}]$ is the nonlinear mechanical stiffness matrix, $[K_{u\phi}]$ and $[K_{\phi u}]$ are nonlinear piezoelectric stiffness matrices, $[K_{\phi\phi}]$ is the nonlinear dielectric stiffness matrix, $[f_i]$ is the external mechanical force vector, $[g_i]$ is the external electrical force vector due to applied charge density respectively. It may be noted that superscript (t) appears in these matrices to denote that the field variables used in the expressions correspond to those of previous time step (t). The superscript $(t+1)$ is for the current time step. The expressions for these matrices are given in Appendix A.3. It may be noted that the nonlinear stiffness and damping matrices depend implicitly upon the solution, i.e., $\{u_i\}$ and $\{\phi_i\}$ through the vectors $\{S\}$ and $\{E\}$. In order to derive and assemble the element stiffness and damping matrices, the classical linearization technique (Salinas et al., 1993) has been adapted here. This classical linearization technique of Salinas et al. (1993) for scalar nonlinear equations has been extended to the nonlinear vector expressions.

4.2. Derivation of finite element equations for nonlinear damping formulation

Similar to the case of proportional damping formulation, when the expressions $\{S\}$ for $\{E\}$ are substituted in the variational Eq. (8) and the expression of virtual work for nonlinear damping formulation W_D from Eq. (13) was used, the coupled FE equations with nonlinear terms appearing as force vectors are obtained and are same as that of Eq. (16) except for the expressions of the nonlinear mechanical and electrical force vectors (i.e., $\{f_{m\_i}\}^{(t+1)}_{\text{nonl}}$ and $\{f_{e\_i}\}^{(t+1)}_{\text{nonl}}$). These force vectors at the current time step $(t+1)$ are then splitted into nonlinear stiffness as well as damping matrices after following a classical linearization technique (Salinas et al., 1993). The linearization technique followed in this work is an extension of the classical linearization technique (Salinas et al. (1993) for nonlinear scalar equations) to vector expressions. The details of the procedure are described in Appendix A.4 and the expression for the nonlinear matrices for this case are given in Appendix A.5. The coupled nonlinear FE equation that is obtained from this method is expressed as

$$\begin{aligned}&[M]^{(t+1)}\{\ddot{u}_i\}^{(t+1)} + [C_{\text{mdamp}}]^{(t)}\{\dot{u}_i\}^{(t+1)} + [C_{\text{pdamp}}]^{(t)}\{\dot{\phi}_i\}^{(t+1)} + [K_{uu}]^{(t)}\{u_i\}^{(t+1)} + [K_{u\phi}]^{(t)}\{\phi_i\}^{(t+1)} \\ &\quad + [K_{uud}]^{(t)}\{u_i\}^{(t+1)} + [K_{u\phi d}]^{(t)}\{\phi_i\}^{(t+1)} = \{f_i\}^{(t+1)}\end{aligned} \tag{19}$$

$$\begin{aligned}&[e_{\text{pdamp}}]^{(t)}\{\dot{u}_i\}^{(t+1)} + [e_{\text{ddamp}}]^{(t)}\{\dot{\phi}_i\}^{(t+1)} + [K_{\phi u}]^{(t)}\{u_i\}^{(t+1)} + [K_{\phi\phi}]^{(t)}\{\phi_i\}^{(t+1)} \\ &\quad + [K_{\phi ud}]^{(t)}\{u_i\}^{(t+1)} + [K_{\phi\phi d}]^{(t)}\{\phi_i\}^{(t+1)} = \{g_i\}^{(t+1)}\end{aligned} \tag{20}$$

where $[C_{\mathrm{mdamp}}]$ is the nonlinear mechanical damping matrix, $[C_{\mathrm{pdamp}}]$, $[e_{\mathrm{pdamp}}]$ are the nonlinear piezoelectric damping matrices $[e_{\mathrm{ddamp}}]$, is the nonlinear dielectric damping matrix $[K_{uud}]$, is the additional nonlinear mechanical stiffness matrix $[K_{u\phi d}]$, $[K_{\phi ud}]$, are the additional nonlinear piezoelectric stiffness matrices and $[K_{\phi\phi d}]$ is the additional nonlinear dielectric stiffness matrix respectively due to the nonlinear damping formulation. The superscripts (t) and $(t+1)$ correspond to previous and current time steps respectively. The final coupled FE equations that can be assembled and solved by the Newmark-β (Bathe, 1995) method with iteration are

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}^{(t+1)} \begin{Bmatrix} \ddot{u}_i \\ \ddot{\phi}_i \end{Bmatrix}^{(t+1)} + \begin{bmatrix} C_{\mathrm{mdamp}} & C_{\mathrm{pdamp}} \\ e_{\mathrm{pdamp}} & e_{\mathrm{ddamp}} \end{bmatrix}^{(t)} \begin{Bmatrix} \dot{u}_i \\ \dot{\phi}_i \end{Bmatrix}^{(t+1)} + \begin{bmatrix} K_{uu} + K_{uud} & K_{u\phi} + K_{u\phi d} \\ K_{\phi u} + K_{\phi ud} & K_{\phi\phi} + K_{\phi\phi d} \end{bmatrix}^{(t)} \begin{Bmatrix} u_i \\ \phi_i \end{Bmatrix}^{(t+1)} = \begin{Bmatrix} f_i \\ g_i \end{Bmatrix}^{(t+1)} \tag{21}$$

5. Conclusion

Piezoceramic continua exhibit different types of nonlinearities under weak electric fields (when the system is operating near resonance frequency) such as jump phenomena, dependence of resonance frequency on vibration amplitude etc. In this work, a generalized nonlinear electric enthalpy density function as well the virtual work due to damping incorporating higher order nonlinear terms (quadratic and cubic) in the conservative as well as in the dissipative energy expression of the coupled piezoelectric medium have been formulated. This nonlinear electric enthalpy density function as well as the virtual work due to nonlinear damping has been used to derive the coupled FE equations through variational formulation. The element level equations have been assembled using linearization technique and the global equations obtained are suitable for solution using Newmark-method with iteration. In a companion paper (Samal et al., 2005), numerical results on various typical examples (using different geometries as well as materials) are shown to match very well with the experimental observations.

Acknowledgement

The authors are grateful to the German Academic Exchange Service (DAAD) for providing a fellowship to Mr. M.K. Samal to carry out part of his thesis work at Darmstadt University of Technology, Germany. This work is supported by the Deutsche Forschungs Gemeinschaft vide research grant Ha1060/36. The first two authors gratefully acknowledge the support received from Aeronautics Research and Development Board and Department of Science and Technology, India.

Appendix A

A.1. Derivation of nonlinear electric enthalpy density function for 1-D piezoelectric beam

The linear constitutive equations for the d_{31} effect are given as (von Wagner et al., 2001):

$$T_{xx} = E_c S_{xx} - d_{31} E_c E_z$$
$$D_z = d_{31} E_c S_{xx} + (\varepsilon_{33}^{\mathrm{T}} - d_{31}^2 E_c) E_z$$

where T_{xx} and S_{xx} are the stress and strain respectively in x direction (length of beam). E_c is the elastic modulus of the piezoceramic orthogonal to polarization direction and E_z is the applied electric field in the z-direction. The parameters d_{31} and ε_{33}^{T} correspond to the 31-piezoelectric effect and dielectric constant at constant stress respectively. To include higher order terms of strain dependence, the elastic modulus and d_{31} parameter can be represented as

$$E_c = E_c^{(0)} + E_c^{(1)} S_{xx} + E_c^{(2)} S_{xx}^2$$
$$d_{31} = d_{31}^{(0)} + d_{31}^{(1)} S_{xx} + d_{31}^{(2)} S_{xx}^2$$

Assuming a linear relationship between D_z and E_z and retaining terms only upto the third order, the nonlinear constitutive equation for T_{xx} and D_z can be written as

$$T_{xx} = E_c^{(0)} S_{xx} + E_c^{(1)} S_{xx}^2 + E_c^{(2)} S_{xx}^3 - \gamma_0 E_z - \gamma_1 S_{xx} E_z - \gamma_2 S_{xx}^2 E_z$$
$$D_z = \gamma_0 S_{xx} + \frac{1}{2}\gamma_1 S_{xx}^2 + \frac{1}{3}\gamma_2 S_{xx}^3 + v_0 E_z$$

where

$$v_0 = \varepsilon_{33}^{T} - (d_{33}^{(0)})^2 E_c^{(0)}$$
$$\gamma_0 = E_c^{(0)} d_{31}^{(0)}$$
$$\gamma_1 = E_c^{(0)} d_{31}^{(1)} + E_c^{(1)} d_{31}^{(0)}$$
$$\gamma_2 = E_c^{(0)} d_{31}^{(2)} + E_c^{(2)} d_{31}^{(0)} + E_c^{(1)} d_{31}^{(1)}$$

One must satisfy the compatibility equations, i.e.,

$$T_{xx} = \frac{\partial H}{\partial S_{xx}} \quad \text{and} \quad D_z = -\frac{\partial H}{\partial E_z}$$

Considering the above expressions, the expression for the nonlinear electric enthalpy function can be derived as

$$H = H_{\text{nonl}} = \frac{1}{2}E_c^{(0)} S_{xx}^2 + \frac{1}{3}E_c^{(1)} S_{xx}^3 + \frac{1}{4}E_c^{(2)} S_{xx}^4 - \gamma_0 S_{xx} E_z - \frac{1}{2}\gamma_1 S_{xx}^2 E_z - \frac{1}{3}\gamma_2 S_{xx}^3 E_z - \frac{1}{2} v_0 E_z^2$$

This expression for the nonlinear electric enthalpy function also satisfies the necessity and sufficient condition for the existence of the electric enthalpy function, i.e.,

$$\frac{\partial^2 H}{\partial S_{xx}\partial E_z} = \frac{\partial T_{xx}}{\partial E_z} = -\frac{\partial D_z}{\partial S_{xx}} = \frac{\partial^2 H}{\partial E_z \partial S_{xx}}$$

The nonlinear electric enthalpy density function has been extended to 3-D piezoelectric continua in the present work.

A.2. Variational form of virtual work for nonlinear damping

$$\delta W_{\mathrm{D}} = \int_V \{\delta S\}^{\mathrm{T}} \begin{bmatrix} [C_d^{(0)}]\{\dot{S}\} - [\gamma_{0d}]\{\dot{E}\} - \frac{2}{3}[C_d^{(1)}]\{S\dot{S}\} + \frac{2}{3}[\mathrm{diag}(\{\dot{S}\})][C_d^{(1)}]\{S\} \\ + \frac{2}{3}[\mathrm{diag}(\{S\})][C_d^{(1)}]\{\dot{S}\} - [\mathrm{diag}(\{S\})][\gamma_{1d}^{(1)}]\{\dot{E}\} - [\mathrm{diag}(\{\dot{S}\})][\gamma_{1d}^{(1)}]\{E\} \\ -[\gamma_{1d}^{(2)}]\{E\dot{E}\} + \frac{3}{4}[C_{d1}^{(2)}]\{S^2\dot{S}\} + \frac{3}{2}[\mathrm{diag}(\{S\dot{S}\})][C_{d1}^{(2)}]^{\mathrm{T}}\{S\} \\ + \frac{3}{4}[\mathrm{diag}(\{S^2\})][C_{d1}^{(2)}]^{\mathrm{T}}\{\dot{S}\} + 2[\mathrm{diag}(\{S\})][C_{d2}^{(2)}]^{\mathrm{T}}\{S\dot{S}\} + [\mathrm{diag}(\dot{S})][C_{d2}^{(2)}]^{\mathrm{T}}\{S^2\} \\ -[\mathrm{diag}(\{S^2\})][\gamma_{2d}^{(1)}]\{\dot{E}\} - 2[\mathrm{diag}(\{S\dot{S}\})][\gamma_{2d}^{(1)}]\{E\} - 2[\mathrm{diag}(\{S\})][\gamma_{2d}^{(2)}]\{E\dot{E}\} \\ -[\mathrm{diag}(\{\dot{S}\})][\gamma_{2d}^{(2)}]\{E^2\} - [\gamma_{2d}^{(3)}]\{E^2\dot{E}\} \end{bmatrix}$$

$$+\{\delta E\}^{\mathrm{T}}\left[\begin{array}{l}[\gamma_{0d}]^{\mathrm{T}}\{\dot{S}\}+\frac{1}{2}[\gamma_{1d}^{(1)}]^{\mathrm{T}}\{\dot{S^2}\}+[\mathrm{diag}(\{\dot{E}\})][\gamma_{1d}^{(2)}]^{\mathrm{T}}\{S\}\\ \quad+[\mathrm{diag}(\{E\})][\gamma_{1d}^{(2)}]^{\mathrm{T}}\{\dot{S}\}+\frac{1}{3}[\gamma_{2d}^{(1)}]^{\mathrm{T}}\{\dot{S^3}\}+[\mathrm{diag}(\{\dot{E}\})][\gamma_{2d}^{(2)}]^{\mathrm{T}}\{S^2\}\\ \quad+[\mathrm{diag}(\{E\})][\gamma_{2d}^{(2)}]^{\mathrm{T}}\{\dot{S^2}\}+[\mathrm{diag}(\{\dot{E^2}\})][\gamma_{2d}^{(3)}]^{\mathrm{T}}\{S\}+[\mathrm{diag}(\{E^2\})][\gamma_{2d}^{(3)}]^{\mathrm{T}}\{\dot{S}\}\\ \quad+[v_{0d}]\{\dot{E}\}+\frac{2}{3}[\mathrm{diag}(\{E\})][v_{1d}]\{\dot{E}\}+\frac{2}{3}[\mathrm{diag}(\{\dot{E}\})][v_{1d}]\{E\}\\ \quad+\frac{1}{3}[v_{1d}]^{\mathrm{T}}\{\dot{E^2}\}+\frac{3}{4}[\mathrm{diag}(\{E^2\})][v_{2d1}]\{\dot{E}\}+\frac{3}{4}[\mathrm{diag}(\{\dot{E^2}\})][v_{2d1}]\{E\}\\ \quad+\frac{1}{4}[v_{2d1}]^{\mathrm{T}}\{\dot{E^3}\}+[\mathrm{diag}(\{\dot{E}\})][v_{2d2}]\{E^2\}+[\mathrm{diag}(\{E\})][v_{2d2}]^{\mathrm{T}}\{\dot{E}^2\}\end{array}\right]\mathrm{d}V$$

where the notation [diag{ }] represents a diagonal matrix with the terms of the diagonal being equal to the corresponding terms of a vector that it is enclosing.

A.3. Finite element matrices of the nonlinear piezoelectric continuum for proportional damping formulation corresponding to Eqs. (17) and (18)

$$[M]=\int_V [N_u]^{\mathrm{T}}\rho[N_u]\,\mathrm{d}V$$

$$[K_{uu}]=\int_V\left[\begin{array}{l}[B_u]^{\mathrm{T}}[C][B_u]+[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S\})][C_{21}][\mathrm{diag}(\{S\})][B_u]\\ \quad+\frac{1}{4}[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S\})][C_{22}][\mathrm{diag}(\{S^2\})][B_u]+\frac{3}{4}[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S^2\})][C_{22}]^{\mathrm{T}}[B_u]\\ \quad+\frac{1}{3}[B_u]^{\mathrm{T}}[C_1][\mathrm{diag}(\{S\})][B_u]+\frac{2}{3}[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S\})][C_1]^{\mathrm{T}}[B_u]\end{array}\right]\mathrm{d}V$$

$$[K_{u\phi}]=\int_V\left[\begin{array}{l}[B_u]^{\mathrm{T}}[C]^{\mathrm{T}}[d]^{\mathrm{T}}[B_\phi]+[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S\})][\gamma_{11}]^{\mathrm{T}}[B_\phi]\\ \quad+\frac{1}{2}[B_u]^{\mathrm{T}}[\gamma_{12}]^{\mathrm{T}}[\mathrm{diag}(\{E\})][B_\phi]+[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S^2\})][\gamma_{21}]^{\mathrm{T}}[B_\phi]\\ \quad+[B_u]^{\mathrm{T}}[\mathrm{diag}(\{S\})][\gamma_{22}]^{\mathrm{T}}[\mathrm{diag}(\{E\})][B_\phi]+\frac{1}{3}[B_u]^{\mathrm{T}}[\gamma_{23}]^{\mathrm{T}}[\mathrm{diag}(\{E^2\})][B_\phi]\end{array}\right]\mathrm{d}V$$

$$[K_{\phi u}]=[K_u\phi]^{\mathrm{T}}$$

$$[K_{\phi\phi}]=-\int_V\left[\begin{array}{l}[B_\phi]^{\mathrm{T}}[v_0][B_\phi]+\frac{1}{3}[B_\phi]^{\mathrm{T}}[v_1][\mathrm{diag}(\{E\})][B_\phi]+\frac{2}{3}[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{E\})][v_1]^{\mathrm{T}}[B_\phi]\\ \quad+\frac{1}{4}[B_\phi]^{\mathrm{T}}[v_{21}][\mathrm{diag}(\{E^2\})][B_\phi]+\frac{3}{4}[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{E^2\})][v_{21}]^{\mathrm{T}}[B_\phi]\\ \quad+[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{E\})][v_{22}][\mathrm{diag}(\{E\})][B_\phi]\end{array}\right]\mathrm{d}V$$

$$[C_{\mathrm{damp}}]=\alpha[M]+\beta\int_V[B_u]^{\mathrm{T}}[C][B_u]$$

$$\{f\}_i=\int_V[N_u]^{\mathrm{T}}\{F_V\}\,\mathrm{d}V+\int_{A_S1}[N_u]^{\mathrm{T}}F_{A_S}dA_S+[N_u]^{\mathrm{T}}\{F_{\mathrm{P}}\}$$

$$\{g_i\}=-\int_{A_{S2}}[N_\phi]^{\mathrm{T}}\varsigma dA_{\mathrm{s}}-[N_\phi]^{\mathrm{T}}Q$$

A.4. Linearization of nonlinear FE equations

The generalized nonlinear mechanical and electrical force vectors $\{f_{m\_i}\}_{\mathrm{nonl}}^{(t+1)}$ and $\{f_{e\_i}\}_{\mathrm{nonl}}^{(t+1)}$ that are presented in the Section 4.2 are for the FE formulation with nonlinear damping. However, these vectors can also be derived for the case of proportional damping formulation. In order to present the methodology of classical linearization (Salinas et al., 1993), only the details of the nonlinear damping case are presented here. These vectors have been derived from the variational Eq. (8) using the virtual work δW_{D} [Appendix A.2] and are written as

$$
\begin{aligned}
(f_{m\_i})^{(t+1)}_{\text{nonl}} = {} & \frac{2}{3}[B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][c_d^{(1)}]^{\mathrm{T}}[B_u]\{U^{(t+1)}\} + \frac{3}{2}[B_u]^{\mathrm{T}}[\text{diag}(\{S\dot{S}\})][c_{d1}^{(2)}]^{\mathrm{T}}[B_u]\{U^{(t+1)}\} \\
& + [B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][c_{d2}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u]\{U^{(t+1)}\} + [B_u]^{\mathrm{T}}[c_d^{(0)}]^{\mathrm{T}}[B_u]\{\dot{U}^{(t+1)}\} \\
& + \frac{2}{3}[B_u]^{\mathrm{T}}[c_d^{(1)}][\text{diag}(\{S\})][B_u]\{\dot{U}^{(t+1)}\} + \frac{2}{3}[B_u]^{\mathrm{T}}[\text{diag}(\{S\})][c_d^{(1)}]^{\mathrm{T}}[B_u]\{\dot{U}^{(t+1)}\} \\
& + \frac{3}{4}[B_u]^{\mathrm{T}}[c_{d1}^{(2)}][\text{diag}(\{S^2\})][B_u]\{\dot{U}^{(t+1)}\} + \frac{3}{4}[B_u]^{\mathrm{T}}[\text{diag}(\{S^2\})][c_{d1}^{(2)}]^{\mathrm{T}}[B_u]\{\dot{U}^{(t+1)}\} \\
& + 2[B_u]^{\mathrm{T}}[\text{diag}(\{S\})][c_{d2}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u]\{\dot{U}^{(t+1)}\} \\
& + [B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][\gamma_{1d}^{(1)}][B_\phi]\{\Phi^{(t+1)}\} + 2[B_u]^{\mathrm{T}}[\text{diag}(\{S\dot{S}\})][\gamma_{2d}^{(1)}][B_\phi]\{\Phi^{(t+1)}\} \\
& + 2[B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][\gamma_{2d}^{(2)}][\text{diag}(\{E\})][B_\phi]\{\Phi^{(t+1)}\} + [B_u]^{\mathrm{T}}[\gamma_{0d}][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
& + [B_u]^{\mathrm{T}}[\text{diag}(\{S\})][\gamma_{1d}^{(1)}][B_\phi]\{\dot{\Phi}^{(t+1)}\} + [B_u]^{\mathrm{T}}[\gamma_{1d}^{(2)}][\text{diag}(\{E\})][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
& + [B_u]^{\mathrm{T}}[\text{diag}(\{S^2\})][\gamma_{2d}^{(1)}][B_\phi]\{\dot{\Phi}^{(t+1)}\} + 2[B_u]^{\mathrm{T}}[\text{diag}(\{S\})][\gamma_{2d}^{(2)}][\text{diag}(\{E\})][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
& + [B_u]^{\mathrm{T}}[\gamma_{2d}^{(3)}][\text{diag}(\{E^2\})][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
(f_{e\_i})^{(t+1)}_{\text{nonl}} = {} & [B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][\gamma_{1d}^{(2)}]^{\mathrm{T}}[B_u]\{U^{(t+1)}\} \\
& + [B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][\gamma_{2d}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u]\{U^{(t+1)}\} \\
& + [B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}^2\})][\gamma_{2d}^{(3)}]^{\mathrm{T}}[B_u]\{U^{(t+1)}\} + [B_\phi]^{\mathrm{T}}[\gamma_{0d}]^{\mathrm{T}}[B_u]\{\dot{U}^{(t+1)}\} \\
& + [B_\phi]^{\mathrm{T}}[\gamma_{1d}^{(1)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u]\{\dot{U}^{(t+1)}\} + [B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][\gamma_{1d}^{(2)}]^{\mathrm{T}}[B_u]\{\dot{U}^{(t+1)}\} \\
& + \frac{1}{3}[B_\phi]^{\mathrm{T}}[\gamma_{2d}^{(1)}]^{\mathrm{T}}[\text{diag}(\{S^2\})][B_u]\{\dot{U}^{(t+1)}\} \\
& + 2[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][\gamma_{2d}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u]\{\dot{U}^{(t+1)}\} \\
& + [B_\phi]^{\mathrm{T}}[\text{diag}(\{E^2\})][\gamma_{2d}^{(3)}]^{\mathrm{T}}[B_u]\{\dot{U}^{(t+1)}\} + \frac{2}{3}[B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][v_{1d}][B_\phi]\{\Phi^{(t+1)}\} \\
& + \frac{3}{4}[B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}^2\})][v_{2d1}][B_\phi]\{\Phi^{(t+1)}\} \\
& + \frac{1}{2}[B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][v_{2d2}][\text{diag}(\{E\})][B_\phi]\{\Phi^{(t+1)}\} \\
& + \frac{1}{2}[B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][v_{2d2}]^{\mathrm{T}}[\text{diag}(\{E\})][B_\phi]\{\Phi^{(t+1)}\} \\
& + [B_\phi]^{\mathrm{T}}[v_{0d}][B_\phi]\{\dot{\Phi}^{(t+1)}\} + \frac{2}{3}[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][v_{1d}][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
& + \frac{2}{3}[B_\phi]^{\mathrm{T}}[v_{1d}]^{\mathrm{T}}[\text{diag}(\{E\})][B_\phi]\{\dot{\Phi}^{(t+1)}\} + \frac{3}{4}[B_\phi]^{\mathrm{T}}[\text{diag}(\{E^2\})][v_{2d1}][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
& + \frac{3}{4}[B_\phi]^{\mathrm{T}}[v_{2d1}]^{\mathrm{T}}[\text{diag}(\{E^2\})][B_\phi]\{\dot{\Phi}^{(t+1)}\} \\
& + 2[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][v_{2d2}]^{\mathrm{T}}[\text{diag}(\{E\})][B_\phi]\{\dot{\Phi}^{(t+1)}\}
\end{aligned}
$$

Using the method of classical linearization, one can separate the accompanying displacement and potential vectors $\{u_i\}^{(t+1)}$, $\{\phi_i\}^{(t+1)}$ and their derivatives $\{\dot{u}_i\}^{(t+1)}, \{\dot{\phi}_i\}^{(t+1)}$ from the expanded forms of the vectors (i.e., nonlinear force vectors are products of stiffness and damping matrices and field variable vectors) and treat the accompanying matrices as those with values at the $(t)^{th}$ time step. The terms of $\{f_{m\_i}\}^{(t+1)}_{\text{nonl}}$ and $\{f_{e\_i}\}^{(t+1)}_{\text{nonl}}$ which contain $\{u_i\}$, $\{\phi_i\}$ explicitly on their RHS are used to form the nonlinear stiffness matrices whereas the terms, which contain the derivatives $\{\dot{u}_i\}, \{\dot{\phi}_i\}$ explicitly on their RHS are used to form the nonlinear damping matrices. If some of the terms do not contain $\{u_i\}$, $\{\phi_i\}$, $\{\dot{u}_i\}$ or $\{\dot{\phi}_i\}$ explicitly on the RHS, it is treated as a force vector in the FE equation of motion. Thus,

$$\{f_{m\_i}\}^{(t+1)}_{\text{nonl}} = [K_{uud}]^t\{u_i\}^{(t+1)} + [K_{u\phi d}]^t\{\phi_i\}^{(t+1)} + [C_{\text{mdamp}}]^t\{\dot{u}_i\}\}^{(t+1)} + [C_{\text{pdamp}}]^t\{\dot{\phi}_i\}^{(t+1)}$$
$$\{f_{e\_i}\}^{(t+1)}_{\text{nonl}} = [K_{\phi ud}]^t\{u_i\}^{(t+1)} + [K_{\phi\phi d}]^t\{\phi_i\}^{(t+1)} + [e_{\text{pdamp}}]^t\{\dot{u}_i\}^{(t+1)} + [C_{\text{ddamp}}]^t\{\dot{\phi}_i\}^{(t+1)}$$

Hence, the element level nonlinear matrices can be assembled with their initial guess values at (t)th time step to form the global matrices and then the system of equations can be solved to get an initial estimate of the solution of the field variables and their derivatives. With these new values, the nonlinear matrices can be updated and again the system of equations solved. This process is continued iteratively till we get a convergence in the solution of the field variables. The corresponding matrices that are presented here are shown in Appendix A.5.

A.5. Finite element matrices of the nonlinear piezoelectric continuum for nonlinear damping formulation corresponding to Eqs. (19) and (20)

$$[C_{\text{mdamp}}] = \int_V \begin{bmatrix} [B_u]^{\mathrm{T}}[C_d^{(0)}][B_u] + \frac{2}{3}[B_u]^{\mathrm{T}}[C_d^{(1)}][\text{diag}(\{S\})][B_u] \\ +\frac{2}{3}[B_u]^{\mathrm{T}}[\text{diag}(\{S\})][C_d^{(1)}]^{\mathrm{T}}[B_u] + \frac{3}{4}[B_u]^{\mathrm{T}}[C_{d1}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S^2\})][B_u] \\ +\frac{3}{4}[B_u]^{\mathrm{T}}[\text{diag}(\{S^2\})][C_{d1}^{(2)}]^{\mathrm{T}}[B_u] \\ +2[B_u]^{\mathrm{T}}[\text{diag}(\{S\})][C_{d2}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u] \end{bmatrix} \mathrm{d}V$$

$$[C_{\text{pdamp}}] = \int_V \begin{bmatrix} [B_u]^{\mathrm{T}}[\gamma_{0d}][B_\phi] + [B_u]^{\mathrm{T}}[\text{diag}(\{S\})][\gamma_{1d}^{(1)}][B_\phi] \\ +[B_u]^{\mathrm{T}}[\gamma_{1d}^{(2)}][\text{diag}(\{E\})][B_\phi] + [B_u]^{\mathrm{T}}[\text{diag}(\{S^2\})][\gamma_{2d}^{(1)}][B_\phi] \\ +2[B_u]^{\mathrm{T}}[\text{diag}(\{S\})][\gamma_{2d}^{(2)}][\text{diag}(\{E\})][B_\phi] \\ +[B_u]^{\mathrm{T}}[\gamma_{2d}^{(3)}][\text{diag}(\{E^2\})][B_\phi] \end{bmatrix} \mathrm{d}V$$

$$[e_{\text{pdamp}}] = \int_V \begin{bmatrix} [B_\phi]^{\mathrm{T}}[\gamma_{0d}]^{\mathrm{T}}[B_u] + [B_\phi]^{\mathrm{T}}[\gamma_{1d}^{(1)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u] \\ +[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][\gamma_{1d}^{(2)}]^{\mathrm{T}}[B_u] + \frac{1}{3}[B_\phi]^{\mathrm{T}}[\gamma_{2d}^{(1)}]^{\mathrm{T}}[\text{diag}(\{S^2\})][B_u] \\ +2[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][\gamma_{2d}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u] \\ +[B_\phi]^{\mathrm{T}}[\text{diag}(\{E^2\})][\gamma_{2d}^{(3)}]^{\mathrm{T}}[B_u] \end{bmatrix} \mathrm{d}V$$

$$[e_{\text{ddamp}}] = \int_V \begin{bmatrix} [B_\phi]^{\mathrm{T}}[\upsilon_{0d}][B_\phi] + \frac{2}{3}[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][\upsilon_{1d}][B_\phi] \\ +\frac{2}{3}[B_\phi]^{\mathrm{T}}[\upsilon_{1d}]^{\mathrm{T}}[\text{diag}(\{E\})][B_\phi] + \frac{3}{4}[B_\phi]^{\mathrm{T}}[\text{diag}(\{E^2\})][\upsilon_{2d1}][B_\phi] \\ +\frac{3}{4}[B_\phi]^{\mathrm{T}}[\upsilon_{2d1}]^{\mathrm{T}}[\text{diag}(\{E^2\})][B_\phi] \\ +2[B_\phi]^{\mathrm{T}}[\text{diag}(\{E\})][\upsilon_{2d2}]^{\mathrm{T}}[\text{diag}(\{E\})][B_\phi] \end{bmatrix} \mathrm{d}V$$

$$[K_{uud}] = \int_V \begin{bmatrix} \frac{2}{3}[B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][C_d^{(1)}]^{\mathrm{T}}[B_u] + \frac{3}{2}[B_u]^{\mathrm{T}}[\text{diag}(\{S\dot{S}\})][C_{d1}^{(2)}]^{\mathrm{T}}[B_u] \\ +[B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][C_{d2}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u] \end{bmatrix} \mathrm{d}V$$

$$[K_{u\phi d}] = \int_V \begin{bmatrix} [B_u]^{\mathrm{T}}[\text{diag}(\{\dot{S}\})][\gamma_{1d}^{(1)}][B_\phi] + 2[B_u]^{\mathrm{T}}[\text{diag}(\{S\dot{S}\})][\gamma_{2d}^{(1)}][B_\phi] \\ +2[B_u]^{\mathrm{T}}[\text{diag}(\{(\dot{S}\})][\gamma_{2d}^{(2)}][\text{diag}(\{E\})][B_\phi] \end{bmatrix} \mathrm{d}V$$

$$[K_{\phi ud}] = \int_V \begin{bmatrix} [B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][\gamma_{1d}^{(2)}]^{\mathrm{T}}[B_u] + [B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}\})][\gamma_{2d}^{(2)}]^{\mathrm{T}}[\text{diag}(\{S\})][B_u] \\ +[B_\phi]^{\mathrm{T}}[\text{diag}(\{\dot{E}^2\})][\gamma_{2d}^{(3)}]^{\mathrm{T}}[B_u] \end{bmatrix} \mathrm{d}V$$

$$[K_{\phi\phi d}] = \int_V \begin{bmatrix} \frac{2}{3}[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{\dot{E}\})][\upsilon_{1d}][B_\phi] + \frac{3}{4}[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{\dot{E}^2\})][\upsilon_{2d1}][B_\phi] \\ +\frac{1}{2}[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{\dot{E}\})][\upsilon_{2d2}][\mathrm{diag}(\{E\})][B_\phi] \\ +\frac{1}{2}[B_\phi]^{\mathrm{T}}[\mathrm{diag}(\{\dot{E}\})][\upsilon_{2d2}]^{\mathrm{T}}[\mathrm{diag}(\{E\})][B_\phi] \end{bmatrix} \mathrm{d}V$$

The expressions for the matrices $[M]$, $[K_{uu}]$, $[K_{u\phi}]$, $[K_{\phi u}]$ and $[K_{\phi\phi}]$ are same as those given in Appendix A.3.

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